ISRAEL JOURNAL OF MATHEMATICS, Vol. 74, No. 1, 1991

SELF-SIMILAR SOLUTIONS FOR A CONVECTION-DIFFUSION EQUATION WITH ABSORPTION IN $\mathbf{R}^N$

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ABSTRACT

We prove the existence of a positive and smooth solution for the following semi-linear elliptic problem:

$$-\Delta f - \frac{x \cdot \nabla f}{2} + |f|^{p-1} f = a \cdot \nabla(|f|^{q-1} f) + \frac{1}{p-1} f \qquad \text{in } \mathbf{R}^N$$

for any $a \in \mathbf{R}^N$, $1 < p < 1 + 2/N$ and $q = (p+1)/2$. This solution decays exponentially as $|x| \to +\infty$. Moreover, if $|a|$ is sufficiently small, this positive and rapidly decaying solution is unique.

The existence of a positive, self-similar solution

$$u(t,x) = t^{-1/(p-1)} f\left(\frac{x}{\sqrt{t}}\right)$$

follows for the following convection–diffusion equation with absorption:

$$u_t - \Delta u + |u|^{p-1} u = a \cdot \nabla(|u|^{q-1} u) \qquad \text{in } \mathbf{R}^N \times (0,\infty).$$

It is also a very singular solution. This solution decays as $|x| \to +\infty$ for any $t > 0$ fixed.

Because of the nonvariational nature of the elliptic problem, a fixed point method is used for proving the existence result. The uniqueness is proved applying the Implicit Function Theorem.

Introduction

We are interested in the existence of so-called "self-similar" solutions of the following reaction diffusion equation with convection:

†The work of the first author has been partially supported by Grant 1273/00003/88 of the University of the Basque Country. The work of the second author has been supported by Grant PB 86-0112-C02-00 of the Dirección General de Investigación Científica y Técnica.

$$u_t - \Delta u + |u|^{p-1}u = a\cdot\nabla(|u|^{q-1}u) \qquad \text{in } \mathbf{R}^N \times (0,\infty) \tag{1}$$

where $p > 1$, $q > 1$, $a \in \mathbf{R}^N$ and "$\cdot$" denotes the scalar product in $\mathbf{R}^N$. Let us briefly recall what these solutions are.

Suppose that u is a solution of (1) and, for $\alpha \in \mathbf{R}$ fixed, consider the following scaling transformation:

$$u_\lambda(t,x) = \lambda^\alpha u(\lambda^2 t, \lambda x) \qquad \forall\lambda > 0, \quad \forall t \in (0,+\infty), \quad \forall x \in \mathbf{R}^N. \tag{2}$$

As a simple calculation shows, if we want u_λ to be also a solution of (1) it is necessary to have

$$q = \frac{p+1}{2} \tag{3}$$

and in this case we must take

$$\alpha = \frac{2}{p-1}. \tag{4}$$

In this situation, any solution u of (1) such that $u_\lambda \equiv u$ for any $\lambda > 0$ is called a forward self-similar solution. We shall refer to them just as self-similar.

It is easy to see that a function u is a self-similar solution of (1) if and only if

$$u(t,x) = t^{-1/(p-1)} f\left(\frac{x}{\sqrt{t}}\right) \tag{5}$$

where $f(x) = u(1,x)$ satisfies the following elliptic equation:

$$-\Delta f - \frac{x\cdot\nabla f}{2} + |f|^{p-1}f = a\cdot\nabla(|f|^{q-1}f) + \frac{1}{p-1}f \qquad \text{in } \mathbf{R}^N. \tag{6}$$

The constant function

$$f \equiv \left(\frac{1}{p-1}\right)^{1/(p-1)} \tag{7}$$

is always a solution of (6).

We are interested in positive solutions decaying to zero as $|x|$ goes to $+\infty$.

First of all let us observe the following. If we integrate by parts in (6) over all of $\mathbf{R}^N$ (this argument is by now formal but, as we shall see below, may be completely justified in the functional framework we shall work) we get

$$\int |f|^{p-1} f\,dx = \left(\frac{1}{p-1} - \frac{N}{2}\right)\int f\,dx$$

(here and in the sequel, when no domain of integration is indicated, it will be understood to be all of $\mathbf{R}^N$).

This identity shows that nonnegative and nonidentically zero solutions for which this integration makes sense cannot exist if $p \geq (N+2)/N$. Therefore we shall only consider the case

$$1 < p < 1 + \frac{2}{N}, \qquad q = \frac{p+1}{2}. \tag{8}$$

Self-similar solutions are of special interest since they often describe the large time behaviour of general solutions.

On the other hand, following H. Brézis, L. A. Peletier and D. Terman in [4], we say that a function v is a very singular solution of (1) if it is a solution of (1) in $t > 0$, x in $\mathbf{R}^N$, smooth in $[0,\infty) \times \mathbf{R}^N$ except at $(0,0)$, $v(0,x) = 0$ for all x in $\mathbf{R}^N \setminus \{0\}$ and is more singular than the heat kernel $K(t,x) = (4\pi t)^{-N/2} e^{-|x|^2/4t}$ (that notion has been introduced in [4] for the equation (1) with $a = 0$). All those properties will be fulfilled by the self-similar solutions of (1) given by (5) with p and q in the range given by (8); for instance, note that

$$K(t,0) = (4\pi t)^{-N/2}$$

while

$$u(t,0) = t^{-1/(p-1)} f(0)$$

and $1/(p-1) > N/2$. Observe also that $\lim_{t\to 0} u(t,\cdot)$ does not exist, even in the weak sense of distributions.

When $a = 0$ (i.e. no convection term) a lot is known about the existence and properties of self-similar solutions.

In [12], A. Gmira and L. Veron proved that the self-similar solution

$$t^{-1/(p-1)} \left(\frac{1}{p-1} \right)^{-1/(p-1)},$$

given by (7), describes the large-time behaviour of the solutions of (1) with initial data such that

$$\lim_{|x|\to\infty} \operatorname{ess} |x|^{2/(p-1)} u_0(x) = +\infty$$

(this is the so-called "strongly nonlinear behaviour").

They also proved that when $p \geq 1 + 2/N$ and the initial data belongs to $L^1(\mathbf{R}^N)$ solutions behave essentially as the heat kernel does ("weakly nonlinear behaviour").

In [4], H. Brézis, L. A. Peletier and D. Terman proved the existence of a radially symmetric positive and exponentially decaying solution of (6). They used a shooting method to solve the corresponding O.D.E. This result was later covered by M. Escobedo and O. Kavian in [8] by using variational methods and in the functional setting we shall use later on. They proved in [9] that, when $1 \leq p < 1 + 2/N$, this positive self-similar solution describes the large-time behaviour of the solutions of (1) with exponentially decaying and nonnegative initial data.

Finally, we just mention that analogous results for the porous media equation have been obtained by S. Kamin and L. A. Peletier in [13],[14].

The equation without absorption term

$$u_t - \Delta u = a \cdot \nabla(|u|^{q-1}u) \qquad \text{in } \mathbf{R}^N \times (0,\infty)$$

has been studied by J. Aguirre, M. Escobedo and E. Zuazua in [1],[2],[3]. In this case again, one can only have a rapidly decaying self-similar solution of constant sign for a fixed value of q, namely $q = (N+1)/N$. It is proved that for every M, the corresponding equation (6)

$$-\Delta f - \frac{x \cdot \nabla f}{2} = a \cdot \nabla(|f|^{1/N} f) + \frac{N}{2} f \qquad \text{in } \mathbf{R}^N$$

has a unique solution with mean equal to M.

Later on, it was shown by M. Escobedo and E. Zuazua in [10],[11], that these solutions describe the large-time behaviour of the solutions of the Cauchy problem $u_t - \Delta u = a \cdot \nabla(|u|^{q-1}u)$ in $\mathbf{R}^N \times (0,\infty)$. Indeed, if the mean of the initial data equals M, the asymptotic behaviour of the corresponding solution is given by the self-similar solution of mean equal to M for any $t > 0$.

Let us finally mention the numerical results of P. L. Sachdev, K. R. C. Nair and V. G. Tikekar [17]. They consider problem (1) with both absorption and convection terms in one space dimension ($N = 1$). Their numerical experiments confirm that the asymptotic behaviour of the solutions of (1) is given by the self-similar solutions of (1) when $1 < p < 3$ and $q = (p+1)/2$.

The aim of this paper is to prove the existence of a positive and rapidly decaying solution for (6) when the absorption and convection terms are both present and for p, q satisfying (8).

In order to state our main result let us introduce some notations and the functional setting where we shall work.

We define $K(x) = \exp(|x|^2/4)$. In addition to the usual Lebesgue and Sobolev spaces in $\mathbf{R}^N$ (whose norms will be denoted by $\|\cdot\|_p$ and $\|\cdot\|_{1,p} \ldots$) we shall use the following weighted spaces:

(a) for $1 \le p < +\infty$ and $r > 0$

$$L^p(K^r) = \left\{ f; \|f\|_{L^p(K^r)} = \left\{ \int |f(x)|^p K^r(x)\, dx \right\}^{1/p} < +\infty \right\};$$

(b) for $m = 1,2,3,\ldots$ and $r > 0$

$$H^m(K^r) = \left\{ f \in L^2(K^r); \|f\|_{H^m(K^r)} = \left\{ \sum_{|\alpha| \le m} \|D^\alpha f\|^2_{L^2(K^r)} \right\}^{1/2} < +\infty \right\}.$$

The differential operator on the left-hand side of (6) can now be written as

$$Lf \equiv -\Delta f - \frac{x \cdot \nabla f}{2} = -\frac{1}{K} \operatorname{div}(K \nabla f).$$

We can now state the main result of this paper:

THEOREM 1. *Assume that* $1 < p < 1 + 2/N$ *and* $q = (p+1)/2$. *Then for any* $a \in \mathbf{R}^N$ *there exists a* C^∞ *and positive solution of* (6) *in* $H^2(K) \cap L^\infty(\mathbf{R}^N)$. *If* $|a|$ *is small enough, this solution is unique in* $H^1(K) \cap L^\infty(\mathbf{R}^N)$.

The solutions of (6) cannot be radially symmetric (the case where $a = 0$, that was treated in [4] and [6], being excepted). Then the shooting methods cannot be applied. On the other hand, as the problem has no variational formulation we are led to use a fixed point method.

We shall seek for solutions of (6) as fixed points of the operator

$$T: H^1(K) \cap L^\infty(\mathbf{R}^N) \to H^1(K) \cap L^\infty(\mathbf{R}^N)$$

defined in the following way: for any $g \in H^1(K) \cap L^\infty(\mathbf{R}^N)$, $Tg = f$ is the solution in $H^1(K) \cap L^\infty(\mathbf{R}^N)$ of the equation

$$Lf + |f|^{p-1} f = a \cdot \nabla(|f|^{q-1} f) + \frac{1}{p-1} g \qquad \text{in } \mathbf{R}^N. \tag{9}$$

Our first task will be to show that T is well defined, that is, to prove the existence and uniqueness of solutions of (9) in $H^1(K) \cap L^\infty(\mathbf{R}^N)$. That will be done by using the Leray–Schauder fixed point theorem and a uniqueness argument already used in [1]. The operator T is also proved to send the nonnegative cone

$$\{f \in H^1(K) \cap L^\infty(\mathbf{R}^N); f \ge 0 \text{ a.e. in } \mathbf{R}^N\}$$

into itself. Then we shall prove that T is compact and apply a fixed point theorem on conical shells (Theorem 20.1 of K. Deimling [7]) (cf., for instance, H. Brézis and R. E. L. Turner [5] and D. G. de Figueiredo, P. L. Lions and R. D. Nussbaum

[6] for other semilinear elliptic equations where this kind of fixed point argument is used).

In order to apply this fixed point argument we need upper and lower bounds for positive solutions of (9). This will be done, on the one hand, by using the "good sign" of the absorption term $|u|^{p-1}u$. On the other hand, we shall use strongly the divergence form of the convection term $a\cdot\nabla(|u|^{q-1}u)$.

Finally, concerning the uniqueness of positive solutions let us notice that we just have a perturbative result. As it is proved in [8] when $a = 0$, equation (6) has a unique positive solution in $H^1(K)$.

The Implicit Function Theorem allows us to prove that we still have uniqueness for $|a|$ sufficiently small.

The rest of the paper is as follows. In section 1 we study the intermediate problem (9). In section 2 we apply the fixed point argument giving first the *a priori* estimates. Finally in section 3 we discuss some extensions of our result to more general equations of the form

$$Lf + \varphi(f) = \operatorname{div}(\psi(f)) + \lambda f \qquad \text{in } \mathbf{R}^N.$$

Let us now briefly mention some results about the weighted spaces defined above which can be found in [8],[15] and that we shall use below:

(a) $\|\nabla f\|_{L^2(K)}$ defines a norm in $H^1(K)$ equivalent to the usual one.

(b) The imbedding $H^1(K) \to L^r(K^{r/2})$ is compact for $2 \le r < 2^* = 2N/(N-2)$ if $N > 2$ and for any $r \ge 1$ finite if $N = 1,2$.

(c) The operator L is self-adjoint in $L^2(K)$. Its domain in $L^2(K)$ is $D(L) = H^2(K)$. Its inverse, as operator from $L^2(K)$ into $L^2(K)$, is compact. The set of eigenvalues of L is

$$\lambda_k = \frac{N+k-1}{2}, \qquad k = 1,2,\ldots.$$

The first eigenvalue is simple and the corresponding eigenspace E_1 is spanned by $\varphi_1 = K^{-1} = \exp(-|x|^2/4)$. Finally L defines an isomorphism between $H^1(K)$ and its dual $(H^1(K))^*$.

1. Construction of the operator T

As has been said in the introduction, in this section we prove that the operator T given by (9) is well defined. That is done by proving the following:

THEOREM 2. *Let $p > 1$ and $q > 1$ be given. Then for any $g \in L^2(K) \cap L^\infty(\mathbf{R}^N)$ and a in $\mathbf{R}^N$ there is a unique solution $f \in H^2(K) \cap L^\infty(\mathbf{R}^N)$ of* (9).

PROOF OF THEOREM 2. We use the same arguments as in the proof of the Theorem of [1]. Therefore some of the steps will only be sketched. We consider only the case $N \geq 3$ since the same kind of arguments applies to the cases $N \leq 2$.

The proof is divided in two parts: Existence and uniqueness.

1. Uniqueness. Assume that there exists two different solutions of (9). Integrating over $\mathbf{R}^N$ we deduce that

$$\int \left(|f_i|^{p-1} f_i + \frac{N}{2} f_i \right) dx = \frac{1}{p-1} \int g, \qquad i = 1,2.$$

Therefore the set $\Omega = \{x \in \mathbf{R}^N; f_1(x) > f_2(x)\}$ has positive measure as well as $\Omega^c = \{x \in \mathbf{R}^N; f_1(x) \leq f_2(x)\}$.

The function $f_1 - f_2$ satisfies

$$L(f_1 - f_2) + (|f_1|^{p-1} f_1 - |f_2|^{p-1} f_2) = a \cdot \nabla(|f_1|^{q-1} f_1 - |f_2|^{q-1} f_2).$$

Integrating this equation over Ω we deduce that

$$\int_\Omega \Delta(f_1 - f_2)\, dx = \frac{N}{2} \int_\Omega (f_1 - f_2)\, dx + \int_\Omega (|f_1|^{p-1} f_1 - |f_2|^{p-1} f_2)\, dx > 0.$$

But, on the other hand, we know from [1],[3] that

$$\int_{\{\tilde{f} > 0\}} \Delta \tilde{f}\, dx \leq 0 \qquad \forall \tilde{f} \in L^1_{\mathrm{loc}}(\mathbf{R}^N), \quad \Delta \tilde{f} \in L^1(\mathbf{R}^N),$$

which contradicts (16).

REMARK 1. The same proof works for proving the uniqueness of solutions for the equation

$$Lf + \Psi_1(f) = a \cdot \nabla(\Psi_2(f)) + \frac{1}{p-1} g$$

where Ψ_1 and Ψ_2 are $C^1(\mathbf{R})$ functions, and Ψ_1 is monotone increasing. That will be used below in the proof of the existence.

2. Existence. First of all let us remark the following. Every C^2 solution of (9) that attains its maximum and minimum satisfies

$$\|f\|_\infty \leq \left(\frac{1}{p-1} \|g\|_\infty \right)^{1/p}. \tag{10}$$

It is clear that in general, when $g \in L^2(K) \cap L^\infty(\mathbf{R}^N)$, the solution f of (9) is not of class C^2. Nevertheless, (10) can be extended by an approximation argument to these solutions. It is therefore necessary to introduce the following truncated equation:

$$Lf + \Psi_1(f) = a \cdot \nabla(\Psi_2(f)) + \frac{1}{p-1} g \qquad \text{in } \mathbf{R}^N \tag{11}$$

where:

(i) Ψ_1 and Ψ_2 belong to $C^{1,\alpha}(\mathbf{R})$ for some $\alpha > 0$;

(ii) Ψ_1 and Ψ_2 are globally Lipschitz, i.e. $\Psi_1', \Psi_2' \in L^\infty(\mathbf{R})$;

(iii) $\Psi_1(s) = |s|^{p-1}s$, $\Psi_2(s) = |s|^{q-1}s$ if

$$|s| \le 1 + \left(\frac{1}{p-1} \|g\|_\infty\right)^{1/p};$$

(iv) Ψ_1 and Ψ_2 are nondecreasing in $\mathbf{R}$ and constants for $|s|$ large enough;

(v) $\Psi_i(-s) = -\Psi_i(s)$ $\forall s \in \mathbf{R}$ for $i = 1,2$.

In order to solve (11) we define the nonlinear operator

$$S : H^1(K) \to H^1(K)$$

such that for any $h \in H^1(K)$, $Sh = f$ is the unique solution of

$$Lf + \Psi_1(f) = a \cdot \nabla(\Psi_2(h)) + \frac{1}{p-1} g \qquad \text{in } \mathbf{R}^N. \tag{12}$$

This last equation is easily solved minimizing the functional

$$\frac{1}{2}\int |\nabla f|^2 K + \int \Phi(f) K - \int \left(a \cdot \nabla(\Psi_2(h)) + \frac{1}{p-1} g\right) f K$$

on the space $H^1(K)$, where $\Phi(s) = \int_0^s \Psi_1(\tau)\, d\tau$.

The uniqueness of the solution of (12) is obvious. On the other hand, it is easy to see that S is continuous from $H^1(K)$ to $H^1(K)$.

Now observe that since $h, f \in H^1(K)$ by the definition of Ψ_1 and Ψ_2 we have

$$-\Psi_1(f) + a \cdot \nabla \Psi_2(h) + \frac{1}{p-1} g \in L^2(K).$$

This implies that $f \in D(L) \equiv H^2(K)$. Multiplying then the equation (12) by Lf in $L^2(K)$ we easily obtain that S sends bounded sets of $H^1(K)$ into bounded sets of $H^2(K)$. Since the embedding of $H^2(K)$ into $H^1(K)$ is compact, we finally deduce that S is compact from $H^1(K)$ into itself.

In order to apply the Leray-Schauder fixed point theorem to the equation (11) we need *a priori* estimates in $H^1(K)$ for the solutions of the following family of problems:

$$Lf + \Psi_1(f) = \theta a \cdot \nabla(\Psi_2(f)) + \frac{1}{p-1} g \qquad \text{in } \mathbf{R}^N, \quad \theta \in [0,1]. \tag{13}$$

We first obtain an *a priori* estimate in $H^1(\mathbf{R}^N)$. Just by multiplying the equation (13) by f in $L^2(\mathbf{R}^N)$ and integrating by parts we obtain

$$\int |\nabla f|^2 \, dx + \frac{N}{4} \int |f|^2 \, dx + \int \Psi_1(f) f \, dx \leq \frac{1}{p-1} \int fg \, dx$$

from which the estimate follows.

Multiplying now (13) by f in $L^2(K)$ we get

$$\|f\|^2_{H^1(K)} + \int \psi_1(f) fK \, dx \leq |a| \, \|\psi_2'(f)\|_\infty \int |\nabla f| \, |f| K \, dx \tag{14}$$

$$+ \frac{1}{p-1} \int |gf| K \, dx \leq \frac{1}{2} \|f\|^2_{H^1(K)} + C(a,g) \{ \|f\|^2_{L^2(K)} + 1 \}$$

with $C(a,g) > 0$ depending continuously on a.

Let us recall now the following inequality proved in [8] (Corollary 1.11):

$$\forall \epsilon > 0, \quad \exists C_\epsilon > 0; \quad \|f\|_{L^2(K)} \leq \epsilon \|f\|_{H^1(K)} + C_\epsilon \|f\|_{H^1(\mathbf{R}^N)}. \tag{15}$$

From (14) and (15) we have, for some positive constant C,

$$\|f\|_{H^1(K)} \leq C \|g\|_{L^2(K)}$$

and then, by the Leray-Schauder theorem, we deduce the existence of a solution of (11).

In order to prove that in fact f solves the equation (9), it is sufficient to see that

$$\|f\|_\infty \leq \left(\frac{1}{p-1} \|g\|_\infty \right)^{1/p}.$$

That can be made by an easy approximation argument. Let be $(g_n) \subset C_0^\infty(\mathbf{R}^N)$ such that $g_n \to g$ in $L^2(K)$.

Solving the corresponding problem

$$Lf_n + \Psi_1(f_n) = a \cdot \nabla(\Psi_2(f_n)) + \frac{1}{p-1} g_n \qquad \text{in } \mathbf{R}^N$$

as above we get a sequence of functions (f_n). Since (g_n) converges in $L^2(K)$, (f_n) is bounded in $H^2(K)$ and then, by the compactness of S, (f_n) is compact in $H^1(K)$. Therefore, by the uniqueness of solutions of (11) (see Remark 1), one has

$$f_n \to f \qquad \text{in } H^1(K) \text{ and a.e. in } \mathbf{R}^N.$$

Now, as $g_n \in C_0^\infty(\mathbf{R}^N)$ and $\Psi_1, \Psi_2 \in C^{1,\alpha}$, a simple bootstrap argument shows that $f_n \in C^2(\mathbf{R}^N)$ and decays at infinity (cf. [3], Prop. 6, for a similar argument). Therefore,

$$\|\Psi_1(f_n)\|_\infty \leq \frac{1}{p-1}\|g_n\|_\infty, \qquad \forall n \geq 1.$$

By construction of Ψ_1 this implies that

$$\|f_n\|_\infty \leq \left(\frac{1}{p-1}\|g_n\|_\infty\right)^{1/p} \tag{16}$$

and then the same inequality holds for f by passing to the limit almost everywhere.

The proof of Theorem 2 is now complete.

REMARK 2. Multiplying in (9) by f^- and integrating over all of $\mathbf{R}^N$ we deduce that

$$g \geq 0 \Rightarrow f \geq 0. \tag{17}$$

REMARK 3. A standard bootstrap argument (cf. [3]) shows that the solution f of (9) satisfies

$$f \in W^{2,r}(\mathbf{R}^N) \qquad \forall r \in [1,\infty). \tag{18}$$

Moreover, T is continuous from $L^2(K) \cap L^\infty(\mathbf{R}^N)$ into $W^{2,r}(\mathbf{R}^N)$ for every $r \in [1,\infty)$.

2. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. This is done by applying the following fixed point theorem in conical shells of K. Deimling [7] to the operator T.

THEOREM ([7], Theorem 20.1, p. 239). *Let X be a Banach space with norm $\|\cdot\|$, $\Gamma \subset X$ a closed cone and $F: \{x \in \Gamma; \|x\| \leq R\} \to \Gamma$ a γ-condensing operator. Suppose that*:

(i) $Fx \neq \lambda x$ *for* $\|x\| = R$, $\lambda > 1$;

(ii) $\exists r \in (0,R)$, $e \in \Gamma$ *such that* $x - Fx \neq \lambda e$ *for* $\|x\| = r$ *and* $\lambda > 0$.

Then F has a fixed point in $\{x \in \Gamma;\ r \leq \|x\| \leq R\}$.

We apply this theorem with $F = T$, $X = H^1(K) \cap L^\infty(\mathbf{R}^N)$ endowed with the norm

$$\|f\| = \|f\|_{H^1(K)} + \|f\|_{L^\infty(\mathbf{R}^N)}$$

and Γ the nonnegative cone of $H^1(K) \cap L^\infty(\mathbf{R}^N)$.

Using the *a priori* estimates of section 1 and the uniqueness of solution for equation (9) it is easy to see that T is continuous from $H^1(K) \cap L^\infty(\mathbf{R}^N)$ into $H^2(K) \cap W^{2,r}(\mathbf{R}^N)$ for every $r \in (1,\infty)$. Since the embedding

$$H^2(K) \cap W^{2,r}(\mathbf{R}^N) \subset H^1(K) \cap L^\infty(\mathbf{R}^N)$$

is compact if $r > N/2$, we deduce that T is compact and thus γ-condensing, from $H^1(K) \cap L^\infty(\mathbf{R}^N)$ into $H^1(K) \cap L^\infty(\mathbf{R}^N)$.

We check now that hypothesis (i) holds. For this we rewrite $Tf = \lambda f$ as

$$Lf + \lambda^{p-1}|f|^{p-1}f = a\lambda^{q-1}\nabla(|f|^{q-1}f) + \frac{1}{\lambda(p-1)}f. \tag{19}$$

Integrating (19) over all of $\mathbf{R}^N$ we deduce

$$\lambda^{p-1}\int |f|^{p-1}f = \left(\frac{1}{\lambda(p-1)} - \frac{N}{2}\right)\int f.$$

Therefore, there exists no solution f of (19) such that $f \geq 0$ and $f \neq 0$ if $\lambda \geq 2/(N(p-1))$. It is then sufficient to prove *a priori* estimates for

$$\lambda \in \left[1, \frac{2}{N(p-1)}\right].$$

From Theorem 2 and Remark 3 we deduce that every solution $f \in H^1(K) \cap L^\infty(\mathbf{R}^N)$ of (19) belongs to $H^2(K) \cap W^{2,r}(\mathbf{R}^N)$ for every $r \in [1,\infty)$. On the other hand, the L^∞ bound of section 1 shows that

$$\lambda^{p-1}\|f\|_\infty^p \leq \frac{1}{\lambda(p-1)}\|f\|_\infty$$

and therefore

$$\|f\|_\infty \leq \left(\frac{1}{\lambda^p(p-1)}\right)^{1/(p-1)} \leq \left(\frac{1}{p-1}\right)^{1/(p-1)}. \tag{20}$$

Multiplying now (19) by f in $L^2(K)$ and using the following variant of inequality (15) (see [8]):

$$\forall \epsilon > 0, \quad \exists C_\epsilon > 0 : \|f\|_{L^2(K)} \leq \epsilon \|f\|_{H^1(K)} + C_\epsilon \|f\|_{L^\infty(\mathbf{R}^N)} \tag{21}$$

and the estimate (20), we easily obtain the $H^1(K)$ estimate.

That estimate in $H^1(K) \cap L^\infty(\mathbf{R}^N)$ shows that the hypothesis (i) is fulfilled for $\mathbf{R}$ large enough.

Let us now check that (ii) holds. Choosing $e = \varphi_1$, the equation $Tf = f - \lambda\varphi_1$ reads

$$\begin{aligned} Lf + |f - \lambda\varphi_1|^{p-1}(f - \lambda\varphi_1) &= a \cdot \nabla(|f - \lambda\varphi_1|^{q-1}(f - \lambda\varphi_1)) \\ &\quad + \frac{1}{p-1} f + \lambda \frac{N}{2} \varphi_1 \qquad \text{in } \mathbf{R}^N. \end{aligned} \tag{22}$$

The question is now to prove an *a priori* lower bound for the solutions of (22) and $\lambda > 0$. Integrating (22) on $\mathbf{R}^N$ we get

$$\int \left(\frac{N}{2} - \frac{1}{p-1} + |f - \lambda\varphi_1|^{p-1} \right) f = \lambda \int \left(\frac{N}{2} + |f - \lambda\varphi_1|^{p-1} \right) \varphi_1 > 0. \tag{23}$$

But now note that, as $\lambda \to 0$ and $\|f\|_\infty \to 0$, we have

$$\frac{N}{2} - \frac{1}{p-1} + |f - \lambda\varphi_1|^{p-1} \to \frac{N}{2} - \frac{1}{p-1} < 0$$

which contradicts (23).

This gives an L^∞ lower bound for solutions of (19) when λ is small enough. More precisely we get that

$$\|f\|_\infty \geq \frac{1}{2} \left(\frac{1}{p-1} - \frac{N}{2} \right)^{1/(p-1)} \tag{24}$$

whenever we have

$$\lambda \leq \lambda_0 = \frac{1}{2} \left(\frac{1}{p-1} - \frac{N}{2} \right)^{1/(p-1)}. \tag{25}$$

We shall prove now an estimate when $\lambda \geq \lambda_0$. If we assume $\|f\|_\infty \leq \epsilon$, then

$$\begin{cases} |f - \lambda\varphi_1|^{p-1} \leq \epsilon^{p-1} + \lambda^{p-1}\varphi_1^{p-1}, \\ |\lambda\varphi_1|^{p-1} - \epsilon^{p-1} \leq |f - \lambda\varphi_1|. \end{cases} \tag{26}$$

Choosing ϵ such that

$$\epsilon \le \min\left\{\left(\frac{1}{p-1} - \frac{N}{2}\right)^{1/(p-1)}, \left(\frac{N}{2}\right)^{1/(p-1)}\right\} \tag{27}$$

we obtain, from (26) and (27),

$$\int\left(\frac{N}{2} - \frac{1}{p-1} + |f - \lambda\varphi_1|^{p-1}\right) f\,dx$$
$$\le \int\left(\frac{N}{2} - \frac{1}{p-1} + \epsilon^{p-1}\right) f + \lambda^{p-1}\int |\varphi_1|^{p-1} f\,dx \le \lambda^{p-1}\int |\varphi_1|^{p-1} f\,dx$$

and

$$\lambda\int\left(\frac{N}{2} + |f - \lambda\varphi_1|^{p-1}\right)\varphi_1 \ge \lambda\left(\frac{N}{2} - \epsilon^{p-1}\right)\int \varphi_1 + \lambda^p \int \varphi_1^p\,dx.$$

Then, from (23) we obtain

$$\int \varphi_1^{p-1} f\,dx \ge \lambda \int \varphi_1^p\,dx \ge \lambda_0 \int \varphi_1^p\,dx. \tag{28}$$

But this inequality gives an $H^1(K)$ lower bound for f since, for some $C > 0$,

$$\|f\|_{H^1(K)} \ge C\|f\|_{L^1(\mathbf{R}^N)} \ge C\int \varphi_1^{p-1} f\,dx \ge C_0 \int \varphi_1^p\,dx > 0.$$

Therefore, all the hypotheses of the fixed point theorem are fulfilled and we obtain a solution f of (1) in the nonnegative cone of $H^1(K) \cap L^\infty(\mathbf{R}^N)$.

Moreover, as in [3], a bootstrap argument shows that in fact this solution f is a classical solution and satisfies

$$f \in W^{2,r}(\mathbf{R}^N) \qquad \forall r \in [1,+\infty).$$

Finally, applying the Hopf's maximum principle (cf. [16], Theorem 5, page 61) over balls we obtain that f is strictly positive on $\mathbf{R}^N$.

3. Uniqueness

We prove in this section the following uniqueness result for the equation (6):

THEOREM 3. *For any $p \in (1, 1 + 2/N)$ and $q = (p+1)/2$ there exists $\delta > 0$ such that, for any $a \in \mathbf{R}^N$ so that $|a| < \delta$, the equation* (6) *has a unique positive solution in $H^1(K) \cap L^\infty(\mathbf{R}^N)$.*

PROOF OF THEOREM 3. The proof is based on the application of the Implicit Function Theorem to a suitable operator.

More precisely, it has already been proved that a function f is a solution of the equation (6) in $H^1(K) \cap L^\infty(\mathbf{R}^N)$ if and only if f satisfies the equation (11). Then let us define the operator:

$$\mathbf{R}^N \times H^1(K) \to (H^1(K))^*,$$

$$(a,h) \to F(a,h) = Lh + \Psi_1(h) - \frac{h}{p-1} + b\cdot\nabla(\Psi_2(h)). \tag{29}$$

It is easy to see that the Frechet derivative of F with respect to h in a point (b,g), denoted $\partial_h F(b,g)$, is given by

$$\forall\varphi \in H^1(K), \quad \partial_h F(b,g)(\varphi) = L\varphi - \frac{1}{p-1}\varphi + \Psi_1'(g)\varphi + b\cdot\nabla(\Psi_2'(g)\varphi).$$

By the construction of Ψ_1 and Ψ_2, it is clear that for any pair (b,g) in $\mathbf{R}^N \times H^1(K)$, $\partial_h F(b,g)$ is linear and continuous from $H^1(K)$ onto $(H^1(K))^*$. Finally, by using the properties of the functions Ψ_1 and Ψ_2 we check easily that $\partial_h F$ is continuous from $\mathbf{R}^N \times H^1(K)$ into $L(H^1(K),(H^1(K))^*)$. Therefore, F is continuously differentiable with respect to h.

On the other hand, it was proved in [8] that for $a = 0$, the equation (6) has a unique positive solution $\bar{f}$ in $H^1(K)$ whenever $1 < p < 1 + 2/N$. In order to apply the Implicit Function Theorem to F at the point $(0,\bar{f})$ we just need to prove the following:

LEMMA 1. *Let p, q and $\bar{f}$ be as above. Then the operator*

$$\partial_h F(0,\bar{f}) : H^1(K) \to (H^1(K))^*,$$

$$v \to Lv - \frac{1}{p-1} v + \Psi_1'(\bar{f})v,$$

is an isomorphism.

PROOF OF LEMMA 1. First of all, as $\bar{f}$ belongs to $H^1(K) \cap L^\infty(\mathbf{R}^N)$ and solves (6) (with $a = 0$) we have $\Psi_1(\bar{f}) \equiv (\bar{f})^p$ and $\Psi_1'(\bar{f}) = p(\bar{f})^{p-1}$. Now, let us define

$$S_{\bar{f}} = L + p(\bar{f})^{p-1}I \quad \text{and} \quad R_{\bar{f}} = L + (\bar{f})^{p-1}I.$$

Both operators $S_{\bar{f}}$ and $R_{\bar{f}}$ are isomorphisms from $H^1(K)$ into $(H^1(K))^*$. On the other hand, by using the compactness of the imbedding from $H^1(K)$ into $L^2(K)$ we deduce that $S_{\bar{f}}$ and $R_{\bar{f}}$ have compact inverses from $L^2(K)$ into $L^2(K)$.

Let $\Psi_1 \in H^1(K) \cap L^\infty(\mathbf{R}^N)$ be the unique positive eigenfunction associated to the first eigenvalue of $S_{\bar{f}}$ such that

$$\int_{\mathbf{R}^N} |\Psi_1|^2 K = 1.$$

Denoting $\lambda_1(S_{\bar{f}})$ and $\lambda_1(R_{\bar{f}})$ the first eigenvalues of $S_{\bar{f}}$ and $R_{\bar{f}}$ respectively, we get, by Raleigh's formulas,

$$\lambda_1(S_{\bar{f}}) = \frac{1}{2}\int_{\mathbf{R}^N} |\nabla\Psi_1|^2 K + \frac{p}{2}\int_{\mathbf{R}^N} (\bar{f})^{p-1}|\Psi_1|^2 K,$$

$$\lambda_1(R_{\bar{f}}) = \operatorname*{Min}_{\int_{\mathbf{R}^N}|\varphi|^2 K\,dx=1} \left(\frac{1}{2}\int_{\mathbf{R}^N} |\nabla\varphi|^2 K + \frac{1}{2}\int_{\mathbf{R}^N} (\bar{f})^{p-1}|\varphi|^2 K\right)$$
$$\leq \frac{1}{2}\int_{\mathbf{R}^N} |\nabla\Psi_1|^2 K + \frac{1}{2}\int_{\mathbf{R}^N} (\bar{f})^{p-1}|\Psi_1|^2 K$$

and then

$$\lambda_1(R_{\bar{f}}) \leq \lambda_1(S_{\bar{f}}) - \frac{p-1}{2}\int (\bar{f})^{p-1}|\Psi_1|^2 K. \tag{30}$$

Now, since $\bar{f}$ satisfies

$$L\bar{f} + |\bar{f}|^{p-1}\bar{f} = \frac{1}{p-1}\bar{f}, \qquad \bar{f} > 0$$

and, by the Krein-Rutman theorem, $R_{\bar{f}}$ has a unique eigenvalue with positive eigenfunction in $H^1(K)$, we deduce that

$$\lambda_1(R_{\bar{f}}) = \frac{1}{p-1}. \tag{31}$$

Combining (30) and (31) we deduce that

$$\partial_h F(0,\bar{f}) = S_{\bar{f}} - \frac{1}{p-1} I$$

is an isomorphism from $H^1(K)$ into $(H^1(K))^*$. Therefore, by the implicit function theorem (see for instance [7], p. 148 Theorem 15.1), there exists a ball $B(0,\epsilon)$ in $\mathbf{R}^N$ and a ball $B(\bar{f},\rho)$ in $H^1(K)$ and one C^1 map

$$T: B(0,\epsilon) \to B(\bar{f},\rho)$$

such that $T(0) = \bar{f}$ and $F(a,f) = 0$ iff $Ta = f$ for every $a \in B(0,\epsilon)$.

In order to conclude the proof of Theorem 3 let us see

LEMMA 2. *Let p and q be as above. Let* $\{a_n\}_n$ *be a sequence in* $\mathbf{R}^N$ *such that* $a_n \to 0$ *as* $n \to +\infty$, *and suppose that*

$$Lf_n + |f_n|^{p-1}f_n = \frac{1}{p-1} f_n + a_n \cdot \nabla(f_n^q), \qquad f_n \in H^1(K), \quad f_n > 0, \quad \text{in } \mathbf{R}^N.$$

Then

$$\lim_{n \to +\infty} \|f_n - \bar{f}\|_{H^1(K)} = 0.$$

PROOF OF LEMMA 2. By the estimates proved in sections 1 and 2 it is clear that $\{f_n\}_n$ is a bounded sequence in $H^2(K) \cap W^{2,r}(\mathbf{R}^N)$ $\forall r \in [1,\infty)$. By compactness we deduce (for a subsequence that, for simplicity, we denote by f_n) the existence of a nonnegative function g in $H^2(K) \cap W^{2,r}(\mathbf{R}^N)$ such that

$$f_n \to g \qquad \text{in } H^1(K) \cap L^\infty(\mathbf{R}^N)$$

and satisfying

$$Lg + g^p = \frac{1}{p-1} g \qquad \text{in } \mathbf{R}^N. \tag{32}$$

Using the lower bound for the solutions of (6) obtained in section 2, we have

$$\|g\|_\infty \geq \left(\frac{1}{p-1} - \frac{N}{2}\right)^{1/(p-1)}.$$

By the uniqueness of positive solutions of (32) in $H^1(K)$ we have $g = \bar{f}$.

The uniqueness of the accumulation point $\bar{f} = g$ ensures that the full sequence f_n converges to $\bar{f}$ in $H^1(K) \cap L^\infty(\mathbf{R}^N)$. The proof of Lemma 2 is concluded.

Therefore, by the above lemma, if $|a| < \epsilon$, for any solution f of (1) we have $(a,f) \in B(0,\epsilon) \times B(\bar{f},\rho)$. Then, (a,f) has to be on the graph of T and so, for $|a|$ small enough, the solution f of (1) is unique.

4. Further remarks

4.1. The arguments of this paper extend to more general equations of the form

$$Lf + \varphi(f) = a \cdot \nabla(\psi(f)) + \lambda f. \tag{33}$$

Under the hypotheses

$$(34)\qquad \begin{cases} \varphi \text{ is nondecreasing; } \varphi \in W^{1,\infty}_{\mathrm{loc}}(R);\ \liminf_{|s|\to\infty} \dfrac{\varphi(s)s}{|s|^2} > \lambda, \\ \varphi(0) = 0, \end{cases}$$

$$(35)\qquad \Psi \in W^{1,\infty}_{\mathrm{loc}}(\mathbf{R}),$$

$$(36)\qquad \lambda > N/2,$$

one proves, in the same way as in sections 1 and 2, that equation (33) has a positive solution in $H^2(K) \cap L^\infty(\mathbf{R}^N)$.

4.2. As has been said in the introduction, the constant $(1/(p-1))^{1/(p-1)}$ is a solution of equation (6). The estimate (20) shows that any solution of (6) belonging to $H^1(K) \cap L^\infty(\mathbf{R}^N)$ is below that constant solution.

In section 3 we have proved the uniqueness of the positive solution of (6) in $H^1(K) \cap L^\infty(\mathbf{R}^N)$ for $|a|$ small. The question of uniqueness of a positive solution for arbitrary a is open.

We note that the uniqueness problem is closely related to the description of the large-time behaviour of the solutions of the Cauchy problem for (1), which is also an open problem.

4.3. Multiplicity of Solutions. It was proved in [8] that if $a = 0$ and

$$(37)\qquad \lambda_k = \frac{N+k-1}{2} < \frac{1}{p-1}$$

defining $d(k) = \sum_{1\le i\le k} \dim \operatorname{Ker}(L - \lambda_k I)$, (6) has at least $2d(k)$ nontrivial solutions. Therefore, applying the Implicit Function Theorem in disjoint neighbourhoods of each of these solutions in the same way as in section 3, we can see that for $|a|$ small enough and p satisfying (37), (6) has at least $2d(k)$ nontrivial solutions in $H^1(K) \cap L^\infty(\mathbf{R}^N)$.

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